FUZZY STABILITY OF A QUARTIC FUNCTIONAL EQUATION VIA TWO METHODS

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Abstract. The main purpose of this article is to institute the generalized Ulam – Hyers stability of a quartic functional equation in fuzzy normed space using two methods.

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1 INTRODUCTION AND PRELIMINARIES

Now, we are commemorating the platinum jubilee for stability of functional equations [1,2,9,10,11,12,13,21,22,23]. The solution and stability of various functional equation in various normed spaces were introduced and discussed in [3,4,18,19,20,24] and reference cited there in.
The solution and stability of the following quartic functional equations

$$\begin{aligned} &g(2x+y+z)+g(2x+y-z)+g(2x-y+z)+g(-2x+y+z)+16g(y)+16g(z) \\ &= 8[g(x+y)+g(x-y)+g(x+z)+g(x-z)]+2[g(y+z)+g(y-z)]+32g(x) \end{aligned} \tag{1.1}$$

$$\begin{aligned} &g(2x+y+z)+g(2x+y-z)+g(2x-y+z)+g(-2x+y+z)+g(2y)+g(2z) \\ &= 8[g(x+y)+g(x-y)+g(x+z)+g(x-z)]+2[g(y+z)+g(y-z)]+32g(x) \end{aligned} \tag{1.2}$$

were investigated by M.Arunkumar [3,4].

Now, we present basic definitions in Fuzzy normed space as given in [5,6,7,8,14,15,16,26].

Definition 1.1 Let X be a real linear space. A function $N : X \times \mathrm{P} \to [0,1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s,t \in \mathrm{P}$,

$(F1)$ $N(x,c) = 0$ for $c \le 0$;

$(F2)$ $x = 0$ if and only if $N(x,c) = 1$ for all $c > 0$;

$(F3)$ $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ if $c \neq 0$;

$(F4)$ $N(x+y, s+t) \ge min\{N(x,s), N(y,t)\}$;

$(F5)$ $N(x,\cdot)$ is a non-decreasing function on P and $lim_{t\to\infty} N(x,t) = 1$;

$(F6)$ for $x \neq 0$, $N(x,\cdot)$ is (upper semi) continuous on P .

The pair (X, N) is called a fuzzy normed linear space. One may regard $N(X,t)$ as the truth-value of the statement the norm of x is less than or equal to the real number t '.

Example 1.2 Let $(X, \|\cdot\|)$ be a normed linear space. Then

$$N(x,t) = \begin{cases} \dfrac{t}{t+\|x\|}, & t > 0,\ x \in X, \\ 0, & t \le 0,\ x \in X \end{cases}$$

is a fuzzy norm on X .

Definition 1.3 Let (X,N) be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N - \lim_{n\to\infty} x_n = x$.

Definition 1.4 A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

Definition 1.5 Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

The main purpose of this article is to institute the generalized Ulam – Hyers stability of a quartic functional equation

$$\begin{aligned} & q(3u_1+2u_2+u_3)+q(3u_1+2u_2-u_3)+q(3u_1-2u_2+u_3)+q(3u_1-2u_2-u_3)+96q(u_2)+48q(u_3) \\ & = 72\left[q(u_1+u_2)+q(u_1-u_2)\right]+18\left[q(u_1+u_3)+q(u_1-u_3)\right]+8\left[q(u_2+u_3)+q(u_2-u_3)\right]+144q(u_1) \end{aligned} \quad (1.3)$$

in fuzzy normed space using direct and fixed point methods.

Now, we will recall the fundamental result in fixed point theory.

Theorem 1.6 [17, 20] **(The Alternative of Fixed Point)** Suppose that for a complete generalized metric space (X,d) and a strictly contractive mapping $T : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either

$(FP1)$ $d(T^n x, T^{n+1}x) = \infty \quad \forall \ n \geq 0,$ or

$(FP2)$ there exists a natural number n_0 such that:

(i) $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;

(ii) The sequence $(T^n x)$ is convergent to a fixed point y^* of T

(iii) y^* is the unique fixed point of T in the set $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;

(iv) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in Y$.

To provide stability results, throughout this article let us presuppose the following.

U be Linear Space; (W, F') be Fuzzy Normed Space; (V, F') be Fuzzy Banach Space .

Now, define a mapping $q : U \longrightarrow V$ by

$$\begin{aligned} Q(u_1,u_2,u_3) &= q(3u_1+2u_2+u_3)+q(3u_1+2u_2-u_3)+q(3u_1-2u_2+u_3)+q(3u_1-2u_2-u_3)+96q(u_2)+48q(u_3) \\ &\quad -72\left[q(u_1+u_2)+q(u_1-u_2)\right]-18\left[q(u_1+u_3)+q(u_1-u_3)\right]-8\left[q(u_2+u_3)+q(u_2-u_3)\right]-144q(u_1) \end{aligned}$$

for all $u_1, u_2, u_3 \in U$.

2. FUZZY STABILITY: DIRECT METHOD ANALYSIS

Theorem 2.1 If $q : U \longrightarrow V$ fulfilling the functional inequality

$$F\left(Q(u_1,u_2,u_3),\delta\right) \geq F'\left(R(u_1,u_2,u_3),\delta\right) \quad (2.1)$$

for all $u_1, u_2, u_3 \in U$ and all $\delta > 0$, where $R : U^3 \longrightarrow W$ be a mapping satisfying the conditions

$$F'\left(R\left(6^{\Diamond}u, 6^{\Diamond}u, 6^{\Diamond}u\right),\delta\right) \geq F'\left(\Delta^{\Diamond} R(u,u,u),\delta\right) \quad (2.2)$$

and

$$\lim_{\beta\to\infty} F'\left(R\left(6^{\Diamond\beta}u, 6^{\Diamond\beta}u, 6^{\Diamond\beta}u\right),\delta\right) = 1 \quad (2.3)$$

for all $u \in U$ and all $\delta > 0$ with $\Diamond = \pm 1$ and $0 < (\Delta/6)^{\Diamond} < 1$. Then there exists one and only quartic function $Q: U \longrightarrow V$ satisfying the functional equation (1.3) and

$$F\left(Q(u) - q(u), \delta\right) \geq \min\left\{F'\left(R(u,u,u), \frac{\delta\left|6^4 - \Delta\right|}{626}\right), F'\left(R(u,0,u), \frac{2\delta\left|6^4 - \Delta\right|}{626}\right), F'\left(R(0,u,0), \frac{623\delta\left|6^4 - \Delta\right|}{626}\right)\right\} \quad (2.4)$$

for all $u \in U$ and all $\delta > 0$.

Proof. First we give proof for $\Diamond = 1$ and the for $\Diamond = -1$ the proof is similar to that of $\Diamond = 1$.

Case 1: $\Diamond = 1$. Replacing (u_1, u_2, u_3) by (u,u,u) in (2.1), we obtain

$$F\left(q(6u) + q(4u) - 97q(u), \delta\right) \geq F'\left(R(u,u,u), \delta\right) \quad (2.5)$$

for all $u \in U$ and all $\delta > 0$. Again replacing (u_1, u_2, u_3) by $(u,0,u)$ in (2.1) and using the definition of FNS, we get

$$F\left(2q(4u) - 16q(4u) - 352q(u), \delta\right) \geq F'\left(R(u,0,u), \delta\right) \quad \text{or}$$

$$F\left(q(4u) - 8q(4u) - 128q(u), \delta\right) \geq F'\left(R(u,0,u), 2\delta\right) \quad (2.6)$$

for all $u \in U$ and all $\delta > 0$. Once again and finally replacing (u_1, u_2, u_3) by $(0,u,0)$ in (2.1) and using the definition of FNS, we have

$$F\left(4q(2u) - 64q(u), \delta\right) \geq F'\left(R(0,u,0), \delta\right)$$

$$F\left(q(2u) - 16q(u), \delta\right) \geq F'\left(R(0,u,0), 4\delta\right) \quad (2.7)$$

for all $u \in U$ and all $\delta > 0$. Combining (2.5), (2.6), (2.7) and using the definition of FNS, we arrive

$$\begin{aligned} F\left(q(6u) - 1296q(u), 626\delta\right) &= F\left(q(6u) + q(4u) - 97q(u) - q(4u) + 8q(4u) + 28q(u) + 89q(2u) - 1424q(u), 626\delta\right) \\ &\geq \min\left\{F\left(q(6u) + q(4u) - 97q(u), \delta\right), F\left(q(4u) - 8q(4u) - 128q(u), 2\delta\right), F\left(89q(2u) - 1424q(u), 623\delta\right)\right\} \\ &\geq \min\left\{F'\left(R(u,u,u), \delta\right), F'\left(R(u,0,u), 2\delta\right), F'\left(R(0,u,0), 623\delta\right)\right\} \end{aligned} \quad (2.8)$$

for all $u \in U$ and all $\delta > 0$. It follows from (2.8) that using the definition of FNS, we achieve

$$F\left(\frac{q(6u)}{6^4} - q(u), \frac{626}{6^4}\delta\right) \geq \min\left\{F'\left(R(u,u,u), \delta\right), F'\left(R(u,0,u), 2\delta\right), F'\left(R(0,u,0), 623\delta\right)\right\} \quad (2.9)$$

for all $u \in U$ and all $\delta > 0$. It is easy to verify from (2.9), (2.2) and using the definition of FNS, we reach

$$F\left(\frac{q(6^{\beta+1}u)}{6^{4(\beta+1)}} - \frac{q(6^{\beta}u)}{6^{\beta}}, \frac{626}{6^4} \cdot \frac{\Delta^{\beta}}{6^{4\beta}}\delta\right) \quad (2.10)$$

$$\geq \min\left\{F'\left(R(u,u,u), \delta\right), F'\left(R(u,0,u), 2\delta\right), F'\left(R(0,u,0), 623\delta\right)\right\}$$

for all $u \in U$ and all $\delta > 0$. From (2.10) one can simple to confirm that

$$\frac{q(6^{\beta}u)}{6^{4\beta}} - q(u) = \sum_{\alpha=0}^{\beta-1} \frac{q(6^{\alpha+1}u)}{6^{4(\alpha+1)}} - \frac{q(6^{\alpha}u)}{6^{\alpha}} \quad (2.11)$$

for all $u \in U$. With the help of (2.11), (2.10) can be remodified as

$$F\left(\frac{q(6^{\beta}u)}{6^{4\beta}} - q(u), \frac{626}{6^4} \cdot \sum_{\alpha=0}^{\beta-1} \frac{\Delta^{\alpha}}{6^{4\alpha}}\delta\right) \geq \min\left\{\bigcup_{\alpha=0}^{\beta-1} F\left(\frac{q(6^{\beta+1}u)}{6^{4(\beta+1)}} - \frac{q(6^{\beta}u)}{6^{\beta}}, \frac{626}{6^4} \cdot \frac{\Delta^{\alpha}}{6^{4\alpha}}\delta\right)\right\}$$

$$\geq \min\left\{F'\left(R(u,u,u), \delta\right), F'\left(R(u,0,u), 2\delta\right), F'\left(R(0,u,0), 623\delta\right)\right\} \quad (2.12)$$

for all $u \in U$ and all $\delta > 0$. If we set u by $6^{\gamma} u$ in (2.12), and using the definition of FNS, we earn that

$$F\left(\frac{q(6^{\beta+\gamma}u)}{6^{4(\beta+\gamma)}}-\frac{q(6^{\gamma}u)}{6^{4\gamma}},\frac{626}{6^4}\cdot\sum_{\alpha=\gamma}^{\gamma+\beta-1}\frac{\Delta^{\alpha}}{6^{4\alpha}}\delta\right) \geq \min\left\{F'\left(R\left(u,u,u\right),\delta\right),F'\left(R\left(u,0,u\right),2\delta\right),F'\left(R\left(0,u,0\right),623\delta\right)\right\} \tag{2.13}$$

for all $u\in U$ and all $\delta>0$. Using (2.2) and the definition of FNS in (2.13), we find the sequence $\left\{\frac{q(6^{\beta}u)}{6^{4\beta}}\right\}$ is a Cauchy sequence and it converges to a point $Q(u)$ in V. Thus, define a mapping $Q:U\longrightarrow V$ by

$$Q(u):F-\lim_{\beta\to\infty}\frac{q(6^{\beta}u)}{6^{4\beta}} \quad or \quad \lim_{\beta\to\infty}F\left(Q(u)-\frac{q(6^{\beta}u)}{6^{4\beta}},\delta\right)=1 \tag{2.14}$$

for all $u\in U$ and all $\delta>0$. Putting $\gamma=0$ and using the definition of FNS in (2.13), we bring out

$$F\left(\frac{q(6^{\beta}u)}{6^{4\beta}}-q(u),\delta\right) \geq \min\left\{F'\left(R\left(u,u,u\right),\frac{\delta}{\frac{626}{6^4}\cdot\sum_{\alpha=0}^{\beta-1}\frac{\Delta^{\alpha}}{6^{4\alpha}}}\right),F'\left(R\left(u,0,u\right),\frac{2\delta}{\frac{626}{6^4}\cdot\sum_{\alpha=0}^{\beta-1}\frac{\Delta^{\alpha}}{6^{4\alpha}}}\right),F'\left(R\left(0,u,0\right),\frac{623\delta}{\frac{626}{6^4}\cdot\sum_{\alpha=0}^{\beta-1}\frac{\Delta^{\alpha}}{6^{4\alpha}}}\right)\right\} \tag{2.15}$$

for all $u\in U$ and all $\delta>0$. Approaching β tends to infinity in (2.15) and using the definition of $Q(u)$, we acquire that

$$F\left(Q(u)-q(u),\delta\right) \geq \min\left\{F'\left(R\left(u,u,u\right),\frac{\delta\left(6^4-\Delta\right)}{626}\right),F'\left(R\left(u,0,u\right),\frac{2\delta\left(6^4-\Delta\right)}{626}\right),F'\left(R\left(0,u,0\right),\frac{623\delta\left(6^4-\Delta\right)}{626}\right)\right\} \tag{2.16}$$

for all $u\in U$ and all $\delta>0$. Now, we want to show that $Q(u)$ satisfies the quartic functional equation (1.3), replacing (u_1,u_2,u_3) by $(6^{\beta}u_1,6^{\beta}u_2,6^{\beta}u_3)$ in (2.1), which yields

$$F\left(\frac{1}{6^{4\beta}}Q(6^{\beta}u_1,6^{\beta}u_2,6^{\beta}u_3),\delta\right)\geq F'\left(R(6^{\beta}u_1,6^{\beta}u_2,6^{\beta}u_3),\delta\right) \tag{2.17}$$

for all $u_1,u_2,u_3\in U$ and all $\delta>0$. At this moment,

$$\begin{aligned}&F\big(Q(3u_1+2u_2+u_3)+Q(3u_1+2u_2-u_3)+Q(3u_1-2u_2+u_3)+Q(3u_1-2u_2-u_3)+96Q(u_2)+48Q(u_3)\\&\quad-72\left[Q(u_1+u_2)+Q(u_1-u_2)\right]-18\left[Q(u_1+u_3)+Q(u_1-u_3)\right]-8\left[Q(u_2+u_3)+Q(u_2-u_3)\right]-144Q(u_1),\delta\big)\\&\geq\min\Big\{F\left(Q(3u_1+2u_2+u_3)-\frac{1}{6^{4\beta}}q(6^{\beta}(3u_1+2u_2+u_3)),\frac{\delta}{14}\right),F\left(Q(3u_1+2u_2-u_3)-\frac{1}{6^{4\beta}}q(6^{\beta}(3u_1+2u_2-u_3)),\frac{\delta}{14}\right),\\&\quad F\left(Q(3u_1-2u_2+u_3)-\frac{1}{6^{4\beta}}q(6^{\beta}(3u_1-2u_2+u_3)),\frac{\delta}{14}\right),F\left(Q(3u_1-2u_2-u_3)-\frac{1}{6^{4\beta}}q(6^{\beta}(3u_1-2u_2-u_3)),\frac{\delta}{14}\right),\\&\quad F\left(96Q(u_2)-\frac{96}{6^{4\beta}}q(6^{\beta}u_2),\frac{\delta}{14}\right),F\left(48Q(u_3)-\frac{48}{6^{4\beta}}q(6^{\beta}u_3),\frac{\delta}{14}\right),F\left(-144Q(u_1)-\frac{144}{6^{4\beta}}q(6^{\beta}u_1),\frac{\delta}{14}\right),\end{aligned}$$

$$F\left(-72Q(u_1+u_2)+\frac{72}{6^{4\beta}}q(6^{\beta}(u_1+u_2)),\frac{\delta}{14}\right),F\left(-72Q(u_1-u_2)+\frac{72}{6^{4\beta}}q(6^{\beta}(u_1-u_2)),\frac{\delta}{14}\right),$$
$$F\left(-18Q(u_1+u_3)+\frac{18}{6^{4\beta}}q(6^{\beta}(u_1+u_3)),\frac{\delta}{14}\right),F\left(-18Q(u_1-u_3)+\frac{18}{6^{4\beta}}q(6^{\beta}(u_1-u_3)),\frac{\delta}{14}\right),$$
$$F\left(-8Q(u_2+u_3)+\frac{8}{6^{4\beta}}q(6^{\beta}(u_2+u_3)),\frac{\delta}{14}\right),F\left(-8Q(u_2-u_3)+\frac{8}{6^{4\beta}}q(6^{\beta}(u_2-u_3)),\frac{\delta}{14}\right),$$
$$F\left(\frac{1}{6^{4\beta}}\left(q(6^{\beta}(3u_1+2u_2+u_3))+q(6^{\beta}(3u_1+2u_2-u_3))+q(6^{\beta}(3u_1-2u_2+u_3))+q(6^{\beta}(3u_1-2u_2-u_3))\right.\right.$$
$$+\frac{96}{6^{4\beta}}q(6^{\beta}u_2)+\frac{48}{6^{4\beta}}q(6^{\beta}u_3)+\frac{144}{6^{4\beta}}q(6^{\beta}u_1)-\frac{72}{6^{4\beta}}\left(q(6^{\beta}(u_1+u_2))+q(6^{\beta}(u_1+u_2))\right)$$
$$\left.\left.-\frac{18}{6^{4\beta}}\left(q(6^{\beta}(u_1+u_3))+q(6^{\beta}(u_1+u_3))\right)-\frac{8}{6^{4\beta}}\left(q(6^{\beta}(u_2+u_3))+q(6^{\beta}(u_2+u_3))\right),\frac{\delta}{14}\right)\right\} \quad (2.18)$$

for all $u_1,u_2,u_3 \in U$ and all $\delta > 0$. Using (2.14) and (2.17) in (2.18) and using the definition of FNS, we land

$$Q(3u_1+2u_2+u_3)+Q(3u_1+2u_2-u_3)+Q(3u_1-2u_2+u_3)+Q(3u_1-2u_2-u_3)+96Q(u_2)+48Q(u_3)$$
$$=72\left[Q(u_1+u_2)+Q(u_1-u_2)\right]+18\left[Q(u_1+u_3)+Q(u_1-u_3)\right]+8\left[Q(u_2+u_3)+Q(u_2-u_3)\right]+144Q(u_1)$$

for all $u_1,u_2,u_3 \in U$. This prove that $Q(u)$ satisfies the quartic functional equation (1.3). Finally, we conclude to prove that $Q(u)$ is unique, assume $Q'(u)$ is another quartic functional equation satisfying (1.3) and (2.4) . Hence

$$F\left(Q(u)-Q'(u),2\delta\right)$$
$$\geq \min\left\{F\left(\frac{Q(6^{\beta}u)}{6^{4\beta}}-\frac{q(6^{\beta}u)}{6^{4\beta}},\delta\right),F\left(\frac{Q'(u)(6^{\beta}u)}{6^{4\beta}}-\frac{q(6^{\beta}u)}{6^{4\beta}},\delta\right)\right\}$$
$$\geq F'\left(R\left(6^{\beta}u,6^{\beta}u,6^{\beta}u\right),\frac{\delta\,6^{4\beta}\left(6^3-\Delta\right)}{\Delta^{\beta}\cdot 626}\right),F'\left(R\left(6_{\beta}u,0,6_{\beta}u\right),\frac{2\delta\,6^{4\beta}\left(6^3-\Delta\right)}{\Delta^{\beta}\cdot 626}\right),F'\left(R\left(0,6_{\beta}u,0\right),\frac{623\delta\,6^{4\beta}\left(6^3-\Delta\right)}{\Delta^{\beta}\cdot 626}\right)$$

for all $u \in U$ and all $\delta > 0$. Applying the convergence, we end up with $Q(u)$ is unique. Thus, the proof of the theorem is complete

Example 2.2 If $q:U \longrightarrow V$ fulfilling the functional inequality

$$F\left(Q\left(u_1,u_2,u_3\right),\delta\right)\geq F'\left(b,\delta\right) \quad (2.19)$$

for all $u_1,u_2,u_3 \in U$ and all $\delta > 0$ where b is a positive integer. Then there exists one and only quartic function $Q:U \longrightarrow V$ satisfying the functional equation (1.3) and

$$F\left(Q\left(u\right)-q(u),\delta\right)\geq F'\left(3b,\delta\left|6^4-1\right|\right) \quad (2.20)$$

for all $u \in U$ and all $\delta > 0$.

Example 2.3 If $q:U \longrightarrow V$ fulfilling the functional inequality

$$F\left(Q\left(u_1,u_2,u_3\right),\delta\right)\geq F'\left(b\sum_{i=1}^{3}|u_i|^{a},\delta\right) \quad (2.21)$$

for all $u_1,u_2,u_3 \in U$ and all $\delta > 0$ where b is a positive integer and $a \neq 4$. Then there exists one and only quartic function $Q:U \longrightarrow V$ satisfying the functional equation (1.3) and

$$F\left(Q(u)-q(u),\delta\right) \geq F'\left(6b\,|u|,\delta\,\left|6^4-6^a\right|\right) \tag{2.22}$$

for all $u \in U$ and all $\delta > 0$.

Example 2.4 If $q: U \longrightarrow V$ fulfilling the functional inequality

$$F\left(Q(u_1,u_2,u_3),\delta\right) \geq F'\left(b\prod_{i=1}^{3}|u_i|^{a_i},\delta\right) \tag{2.23}$$

for all $u_1,u_2,u_3 \in U$ and all $\delta > 0$ where b is a positive integer and $\sum_{i=1}^{3} a_i \neq 4$. Then there exists one and only quartic function $Q: U \longrightarrow V$ satisfying the functional equation (1.3) and

$$F\left(Q(u)-q(u),\delta\right) \geq F'\left(626b\,|u|^{\sum_{i=1}^{3}a_i},\delta\left|6^4-6^{a_1+a_2+a_3}\right|\right) \tag{2.24}$$

for all $u \in U$ and all $\delta > 0$.

3. FUZZY STABILITY: FIXED POINT METHOD ANALYSIS

Theorem 3.1 If $q: U \longrightarrow V$ fulfilling the functional inequality (2.1) for all $u_1,u_2,u_3 \in U$ and all $\delta > 0$, where $R: U^3 \longrightarrow W$ be a mapping satisfying the condition

$$\lim_{\beta\to\infty} F'\left(R\left(\square_\diamond^{\beta}u,\square_\diamond^{\beta}u,\square_\diamond^{\beta}u\right),\delta\right)=1; \text{ where } \square=\begin{cases}6, & \diamond=0;\\ 1/6, & \diamond=1;\end{cases} \tag{3.1}$$

for all $u \in U$ and all $\delta > 0$. If there exists L such that the function

$$F_F'\left(R(u,u,u),626\delta\right)=\min\left\{F'\left(R(u/6,u/6,u/6),\delta\right),F'\left(R(u/6,0,u/6),2\delta\right),F'\left(R(0,u/6,0),623\delta\right)\right\} \tag{3.2}$$

with the property

$$F_F'\left(\frac{1}{\square_\diamond}R\left(\frac{\square_\diamond u}{\square_\diamond},\frac{\square_\diamond u}{\square_\diamond},\frac{\square_\diamond u}{\square_\diamond}\right),\delta\right)=F_F'\left(L\,R(u,u,u),\delta\right) \tag{3.3}$$

for all $u \in U$ and all $\delta > 0$. Then there exists one and only quartic function $Q: U \longrightarrow V$ satisfying the functional equation (1.3) and

$$F\left(Q(u)-q(u),\delta\right) \geq F_F'\left(\frac{L^{1-\diamond}}{1-L}R(u,u,u),\delta\right) \tag{3.4}$$

for all $u \in U$ and all $\delta > 0$.

Proof. Consider a set $\Phi=\left\{p \mid p: U \longrightarrow V, p(0)=0\right\}$. Define a metric d on Φ,

$$d(p,q)=\inf\left\{K \in (0,\infty) \mid F\left(p(x)-q(x),\delta\right) \geq F'\left(K\,R(u,u,u),\delta\right)\right\}.$$

for all $u \in U$ and all $\delta > 0$. It is simple to notice that (Φ,d) is complete. Define $T: \Phi \longrightarrow \Phi$ by $Tq(x)=\frac{1}{\square_\diamond}q(\square_\diamond x)$, for all $x \in U$. For $p,q \in \Phi$, it is effortless to validate that $d(Tp,Tq) \leq Ld(p,q)$. Therefore T is strictly contractive mapping on Φ with Lipschitz constant L. It follows from (2.8) that

$$F\left(q(6u)-1296q(u),626\delta\right) \geq \min\left\{F'\left(R(u,u,u),\delta\right),F'\left(R(u,0,u),2\delta\right),F'\left(R(0,u,0),623\delta\right)\right\} \tag{3.5}$$

for all $u \in U$ and all $\delta > 0$. From (3.5), we arrive

$$F\left(\frac{q(6u)}{6^4}-q(u),626\delta\right)\geq \min\left\{F'\left(\frac{1}{6^4}R(u,u,u),\delta\right),F'\left(\frac{1}{6^4}R(u,0,u),2\delta\right),F'\left(\frac{1}{6^4}R(0,u,0),623\delta\right)\right\} \quad (3.6)$$

for all $u\in U$ and all $\delta>0$. With the help of (3.2), (3.3) and $\lozenge=0$, (3.6) can transformed into

$$F\left(Tq(u)-q(u),\delta\right)\geq F_F^{'}\left(L^1 R(u,u,u),\delta\right)\Rightarrow F\left(Tq(u)-q(u),\delta\right)\geq F_F^{'}\left(L^{1-\lozenge} R(u,u,u),\delta\right) \quad (3.7)$$

for all $u\in U$ and all $\delta>0$. Changing u by $u/6$ in (3.5), we have

$$F\left(q(u)-6^4 q(u/6),626\delta\right)\geq \min\left\{F'\left(R(u/6,u/6,u/6),\delta\right),F'\left(R(u/6,0,u/6),2\delta\right),F'\left(R(0,u/6,0),623\delta\right)\right\} \quad (3.8)$$

for all $u\in U$ and all $\delta>0$. With the help of (3.2), (3.3) and $\lozenge=1$, (3.8) can transformed into

$$F\left(q(u)-Tq(u),\delta\right)\geq F_F^{'}\left(R(u,u,u),\delta\right)\Rightarrow F\left(q(u)-Tq(u),\delta\right)\geq F_F^{'}\left(L^{1-\lozenge} R(u,u,u),\delta\right) \quad (3.9)$$

for all $u\in U$ and all $\delta>0$. From, (3.7) and (3.9), we get $F\left(q(u)-Tq(u),\delta\right)\geq F_F^{'}\left(L^{1-\lozenge} R(u,u,u),\delta\right)$.
Thus, (FP2(i)) of Theorem 1.6 holds. Once again by (FP2(ii)) of Theorem 1.6, there exists a fixed point Q of T in Φ such that $Q(u):F-\lim_{\beta\to\infty}\frac{q(\square_{\lozenge}^{\beta}u)}{\square_{\lozenge}^{4\beta}}$ *or* $\lim_{\beta\to\infty}F\left(Q(u)-\frac{q(\square_{\lozenge}^{\beta}u)}{\square_{\lozenge}^{4\beta}},\delta\right)=1$ for all $u\in U$ and all $\delta>0$. By proceding the identical method in the Theorem 2.1 we can prove the function, $Q:U\to V$ is quartic and it satisfies the functional equation (1.3). Over again by (FP2(iii)) of Theorem 1.6 Q is the one and one function of T in the set $\Psi=\{q\in\Phi\mid d(q,Q)<\infty\}$, such that

$$F\left(q(u)-Q(u),\delta\right)\geq F'\left(K\,R(u,u,u),\delta\right)$$

for all $u\in U$ and all $\delta>0$. All over again by (FP2(iv)) of Theorem 1.6, we desired our result.

Example 3.2 If $q:U\longrightarrow V$ fulfilling the functional inequality

$$F\left(Q(u_1,u_2,u_3),\delta\right)\geq\begin{cases}F'(b,\delta_3);\\ F'\left(b\sum_{i=1}^{3}|u_i|^a,\delta\right);\\ F'\left(b\prod_{i=1}^{3}|u_i|^{a_i},\delta\right);\end{cases} \quad (3.10)$$

for all $u_1,u_2,u_3\in U$ and all $\delta>0$ where b is a positive integer. Then there exists one and only quartic function $Q:U\longrightarrow V$ satisfying the functional equation (1.3) and

$$F\left(Q(u)-q(u),\delta\right)\geq\begin{cases}F'\left(3b,\delta\left|6^4-1\right|\right); & \\ F'\left(6b|u|,\delta\left|6^4-6^a\right|\right); & a\neq 4\\ F'\left(626b|u|^{\sum_{i=1}^{3}a_i},\delta\left|6^4-6^{a_1+a_2+a_3}\right|\right); & a_1+a_2+a_3\neq 4\end{cases} \quad (3.11)$$

for all $u\in U$ and all $\delta>0$.

Proof. If we assume $F\left(R\left(u_1,u_2,u_3\right),\delta\right) \geq \begin{cases} F'(b,\delta); \\ F'\left(b\sum_{i=1}^{3}|u_i|^{a},\delta\right); \\ F'\left(b\prod_{i=1}^{3}|u_i|^{a_i},\delta\right); \end{cases}$ for all $u_1,u_2,u_3 \in U$ and all $\delta > 0$,

then one can see that (3.1) holds. It follows from (3.2) and (3.3), we find that

$$F_F\left(R\left(u,u,u\right),626\delta\right) = \begin{cases} F'(3b,626\delta); \\ F'\left(6b\,|u/6|^{a},626\delta\right); \\ F'\left(b\,|u/6|^{a_1+a_2+a_3},\delta\right); \end{cases}$$

and

$$F'_F\left(\frac{1}{\square_{\lozenge}^{4}}R\left(\frac{u}{\square_{\lozenge}},\frac{u}{\square_{\lozenge}},\frac{u}{\square_{\lozenge}}\right),\delta\right) = \begin{cases} F'\left(\square_{\lozenge}^{-4}3b,626\delta\right); \\ F'\left(\square_{\lozenge}^{a-4}6b\,|u|^{a},626\,\delta\right); \\ F'\left(\square_{\lozenge}^{a_1+a_2+a_3-4}b\,|u|^{a_1+a_2+a_3},\delta\right); \end{cases}$$

for all $u \in U$ and all $\delta > 0$. Hence one bring the inequality (3.11), by considering

	$\lozenge = 0$	$\lozenge = 1$
L	6	$1/6$
L	6^{a-4}	$1/6^{4-a}$
L	$6^{a_1+a_2+a_3-4}$	$1/6^{4-a_1+a_2+a_3}$

Hence the proof is complete.

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